SOLUTIONS TO CHAPTER 1

Problem 1.1

(a) Since the growth rate of a variable equals the time derivative of its log, as shown by equation (1.10) in the text, we can write

$$(1)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln\left[X(t)Y(t)\right]}{dt}.$$

Since the log of the product of two variables equals the sum of their logs, we have

$$(2)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\left[\ln X(t) + \ln Y(t)\right]}{dt} = \frac{d\ln X(t)}{dt} + \frac{d\ln Y(t)}{dt},$$

or simply

$$(3)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{X}(t)}{X(t)} + \frac{\dot{Y}(t)}{Y(t)}.$$

(b) Again, since the growth rate of a variable equals the time derivative of its log, we can write

$$(4)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln\left[X(t)/Y(t)\right]}{dt}.$$

Since the log of the ratio of two variables equals the difference in their logs, we have

$$(5)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\left[\ln X(t) - \ln Y(t)\right]}{dt} = \frac{d\ln X(t)}{dt} - \frac{d\ln Y(t)}{dt},$$

or simply

$$(6)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{X}(t)}{X(t)} - \frac{\dot{Y}(t)}{Y(t)}.$$

(c) We have

$$(7)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln[X(t)^{\alpha}]}{dt}.$$

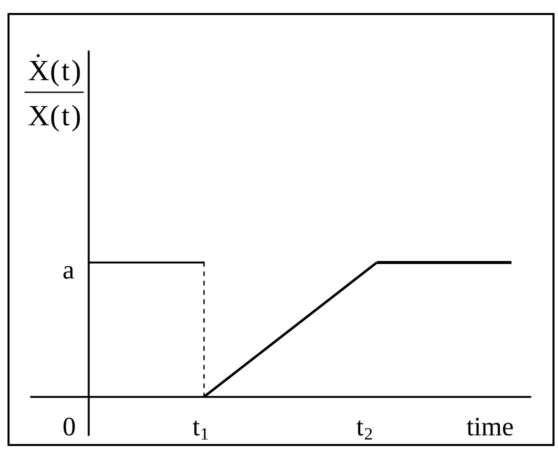
Using the fact that $\ln[X(t)^{\alpha}] = \alpha\ln X(t)$, we have

$$(8)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\left[\alpha\ln X(t)\right]}{dt} = \alpha\frac{d\ln X(t)}{dt} = \alpha\frac{\dot{X}(t)}{X(t)},$$

where we have used the fact that α is a constant.

Problem 1.2

(a) Using the information provided in the question, the path of the growth rate of X, $\dot{X}(t)/X(t)$, is depicted in the figure at right.



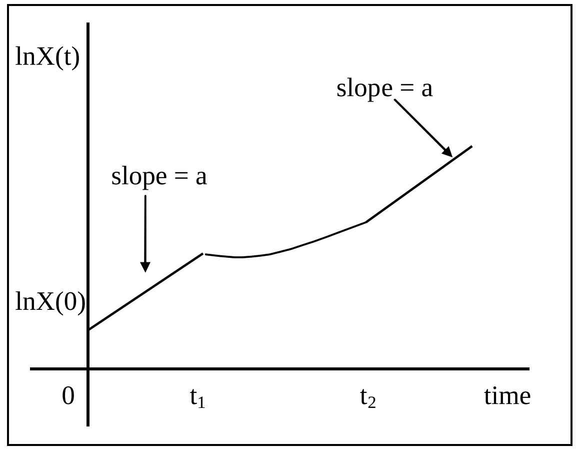
From time 0 to time t_1, the growth rate of X is constant and equal to a > 0. At time t_1, the growth rate of X drops to 0. From time t_1 to time t_2, the growth rate of X rises gradually from 0 to a. Note that we have made the assumption that $\dot{X}(t)/X(t)$ rises at a constant rate from t_1 to t_2. Finally, after time t_2, the growth rate of X is constant and equal to a again.

(b) Note that the slope of lnX(t) plotted against time is equal to the growth rate of X(t). That is, we know

$$\frac{d \ln X(t)}{dt} = \frac{\dot{X}(t)}{X(t)}$$

(See equation (1.10) in the text.)

From time 0 to time t_1 the slope of lnX(t) equals a > 0. The lnX(t) locus has an inflection point at t_1, when the growth rate of X(t) changes discontinuously from a to 0. Between t_1 and t_2, the slope of lnX(t) rises gradually from 0 to a. After time t_2 the slope of lnX(t) is constant and equal to a > 0 again.

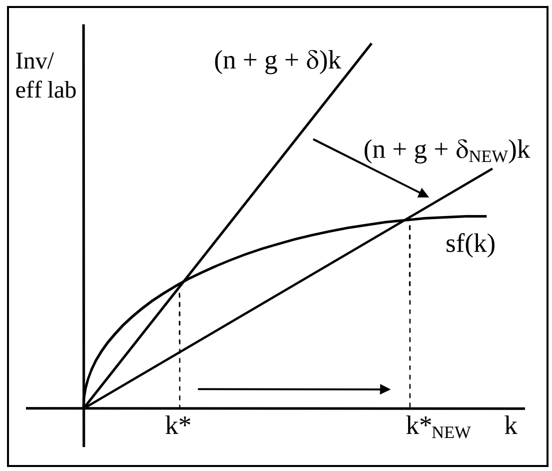


Problem 1.3

(a) The slope of the break-even investment line is given by $(n + g + \delta)$ and thus a fall in the rate of depreciation, δ, decreases the slope of the break-even investment line.

The actual investment curve, sf(k) is unaffected.

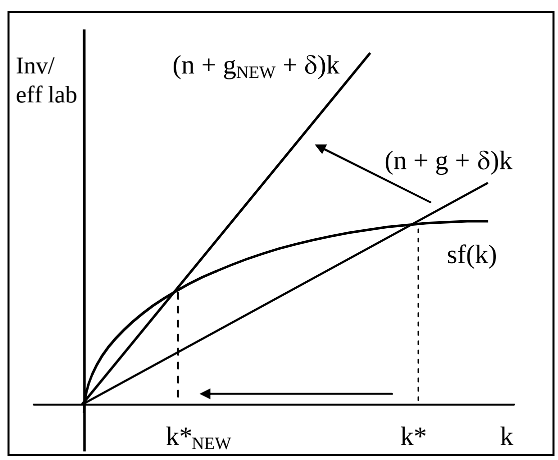
From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor rises from k^* to k^*_{NEW} .



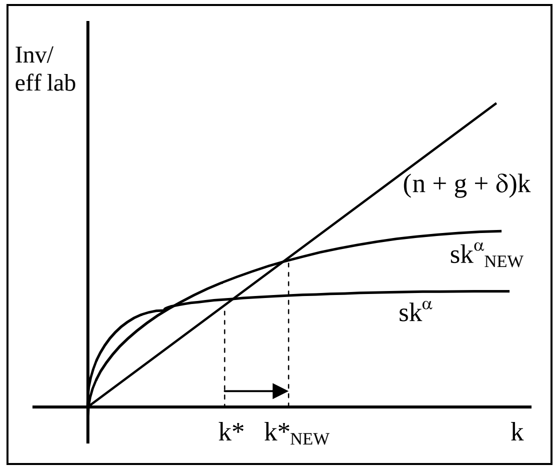
(b) Since the slope of the break-even investment line is given by $(n + g + \delta)$, a rise in the rate of technological progress, g, makes the break-even investment line steeper.

The actual investment curve, sf(k), is unaffected.

From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor falls from k^* to k^*_{NEW} .



(c) The break-even investment line, $(n + g + \delta)k$, is unaffected by the rise in capital's share, α.



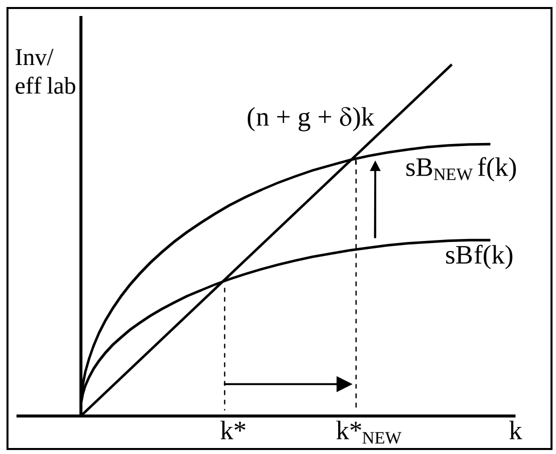
The effect of a change in α on the actual investment curve, sk^{α}, can be determined by examining the derivative $\partial(sk^{\alpha})/\partial\alpha$. It is possible to show that

(1) $$\frac{\partial sk^{\alpha}}{\partial\alpha} = sk^{\alpha}\ln k .$$

For $0 < \alpha < 1$, and for positive values of k, the sign of $\partial(sk^{\alpha})/\partial\alpha$ is determined by the sign of lnk. For $\ln k > 0$, or $k > 1$, $\partial sk^{\alpha}/\partial\alpha > 0$ and so the new actual investment curve lies above the old one. For $\ln k < 0$ or $k < 1$, $\partial sk^{\alpha}/\partial\alpha < 0$ and so the new actual investment curve lies below the old one. At $k = 1$, so that $\ln k = 0$, the new actual investment curve intersects the old one.

In addition, the effect of a rise in α on k* is ambiguous and depends on the relative magnitudes of s and $(n + g + \delta)$. It is possible to show that a rise in capital's share, α, will cause k* to rise if $s > (n + g + \delta)$. This is the case depicted in the figure above.

(d) Suppose we modify the intensive form of the production function to include a non-negative constant, B, so that the actual investment curve is given by $sBf(k)$, $B > 0$.



Then workers exerting more effort, so that output per unit of effective labor is higher than before, can be modeled as an increase in B. This increase in B shifts the actual investment curve up.

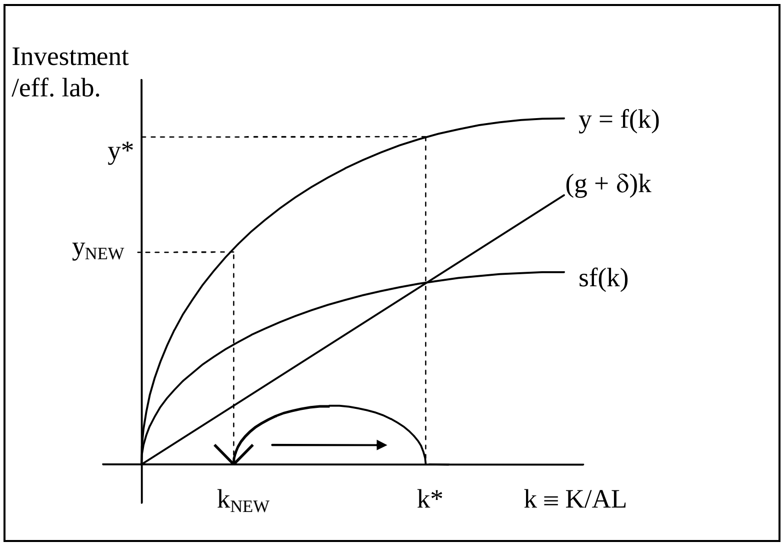
The break-even investment line, $(n + g + \delta)k$, is unaffected.

From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor rises from k* to k^*_{NEW} .

Problem 1.4

(a) At some time, call it t_0 , there is a discrete upward jump in the number of workers. This reduces the amount of capital per unit of effective labor from k* to k_{NEW} . We can see this by simply looking at the definition, $k \equiv K/AL$. An increase in L without a jump in K or A causes k to fall. Since $f'(k) > 0$, this fall in the amount of capital per unit of effective labor reduces the amount of output per unit of effective labor as well. In the figure below, y falls from y* to y_{NEW} .

(b) Now at this lower k_{NEW}, actual investment per unit of effective labor exceeds break-even investment per unit of effective labor. That is, $sf(k_{NEW}) > (g + \delta)k_{NEW}$. The economy is now saving and investing more than enough to offset depreciation and technological progress at this lower k_{NEW}. Thus k begins rising back toward k*. As capital per unit of effective labor begins rising, so does output per unit of effective labor. That is, y begins rising from y_{NEW} back toward y*.



(c) Capital per unit of effective labor will continue to rise until it eventually returns to the original level of k*. At k*, investment per unit of effective labor is again just enough to offset technological progress and depreciation and keep k constant. Since k returns to its original value of k* once the economy again returns to a balanced growth path, output per unit of effective labor also returns to its original value of $y^* = f(k^*)$.

Problem 1.5

(a) The equation describing the evolution of the capital stock per unit of effective labor is given by

(1) $\dot{k} = sf(k) - (n + g + \delta)k$.

Substituting in for the intensive form of the Cobb-Douglas, $f(k) = k^{\alpha}$, yields

(2) $\dot{k} = sk^{\alpha} - (n + g + \delta)k$.

On the balanced growth path, $\dot{k}$ is zero; investment per unit of effective labor is equal to break-even investment per unit of effective labor and so k remains constant. Denoting the balanced-growth-path value of k as k*, we have $sk^{*\alpha} = (n + g + \delta)k^*$. Rearranging to solve for k* yields

(3) $k^* = [s/(n + g + \delta)]^{1/(1-\alpha)}$.

To get the balanced-growth-path value of output per unit of effective labor, substitute equation (3) into the intensive form of the production function, $y = k^{\alpha}$:

(4) $y^* = [s/(n + g + \delta)]^{\alpha/(1-\alpha)}$.

Consumption per unit of effective labor on the balanced growth path is given by $c^* = (1 - s)y^*$. Substituting equation (4) into this expression yields

(5) $c^* = (1 - s)[s/(n + g + \delta)]^{\alpha/(1-\alpha)}$.

(b) By definition, the golden-rule level of the capital stock is that level at which consumption per unit of effective labor is maximized. To derive this level of k, take equation (3), which expresses the balanced-growth-path level of k, and rearrange it to solve for s:

(6) $s = (n + g + \delta)k^{*1-\alpha}$.

Now substitute equation (6) into equation (5):

(7) $c^* = \left[1 - (n + g + \delta)k^{*1-\alpha}\right]\left[(n + g + \delta)k^{*1-\alpha}/(n + g + \delta)\right]^{\alpha/(1-\alpha)}$.

After some straightforward algebraic manipulation, this simplifies to

(8) $c^* = k^{*\alpha} - (n + g + \delta)k^*$.

Equation (8) states that consumption per unit of effective labor is equal to output per unit of effective labor, $k^{*\alpha}$, less actual investment per unit of effective labor. On the balanced growth path, actual investment per unit of effective labor is the same as break-even investment per unit of effective labor, $(n + g + \delta)k^*$.

Now use equation (8) to maximize c* with respect to k*. The first-order condition is given by

(9) $\partial c^*/\partial k^* = \alpha k^{*\alpha-1} - (n + g + \delta) = 0$,

or simply

(10) $\alpha k^{*\alpha-1} = (n + g + \delta)$.

Note that equation (10) is just a specific form of the general condition that implicitly defines the golden-rule level of capital per unit of effective labor, given by $f'(k^*) = (n + g + \delta)$. Equation (10) has a graphical interpretation: it defines the level of k at which the slope of the intensive form of the production function is equal to the slope of the break-even investment line. Solving equation (10) for the golden-rule level of k yields

(11) $k^*_{GR} = [\alpha/(n + g + \delta)]^{1/(1-\alpha)}$.

(c) To get the saving rate that yields the golden-rule level of k, substitute equation (11) into (6):

(12) $s_{GR} = (n + g + \delta)[\alpha/(n + g + \delta)]^{(1-\alpha)/(1-\alpha)}$,

which simplifies to

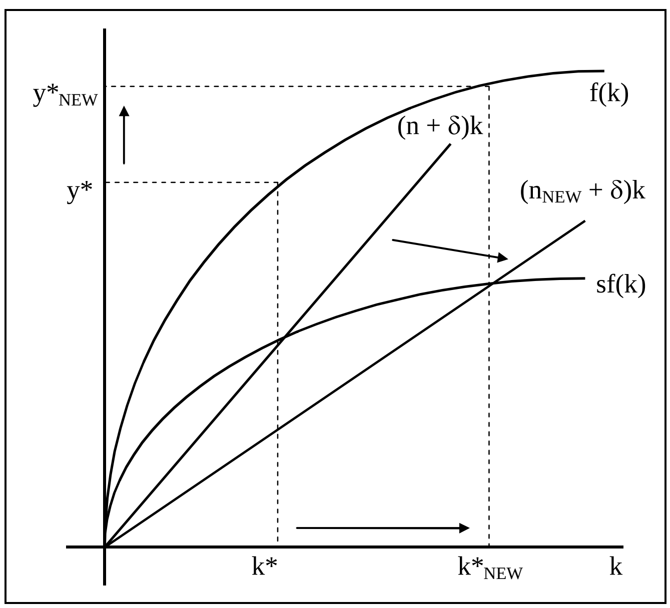
(13) $s_{GR} = \alpha$.

With a Cobb-Douglas production function, the saving rate required to reach the golden rule is equal to the elasticity of output with respect to capital or capital's share in output (if capital earns its marginal product).

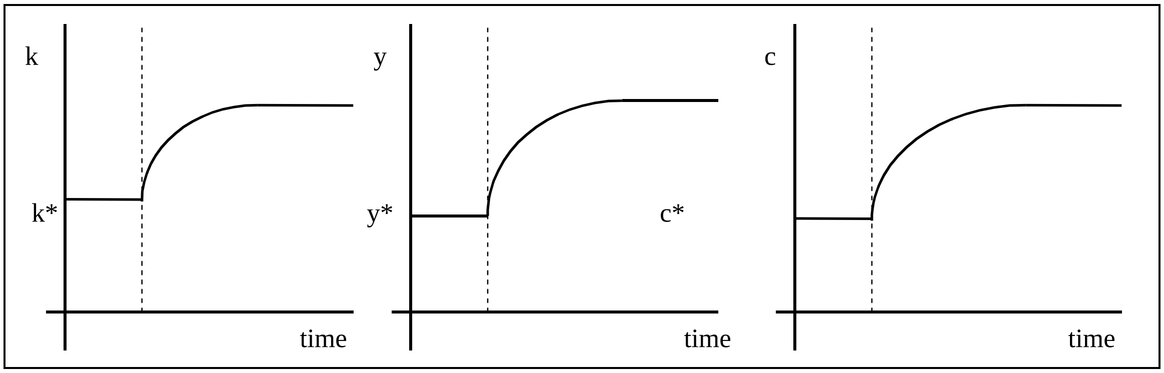
Problem 1.6

(a) Since there is no technological progress, we can carry out the entire analysis in terms of capital and output per worker rather than capital and output per unit of effective labor. With A constant, they behave the same. Thus we can define $y \equiv Y/L$ and $k \equiv K/L$.

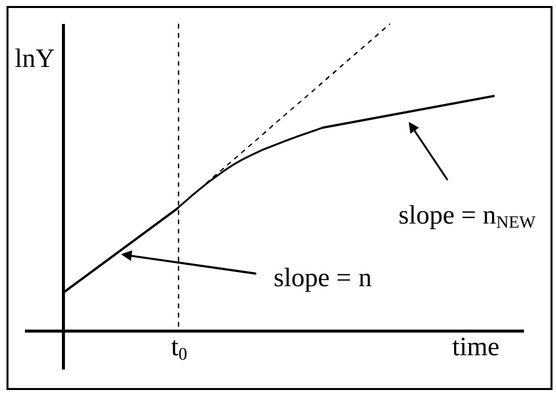
The fall in the population growth rate makes the break-even investment line flatter. In the absence of technological progress, the per unit time change in k, capital per worker, is given by $\dot{k} = sf(k) - (\delta + n)k$. Since $\dot{k}$ was 0 before the decrease in n – the economy was on a balanced growth path – the decrease in n causes $\dot{k}$ to become positive. At k*, actual investment per worker, sf(k*), now exceeds break-even investment per worker, $(n_{NEW} + \delta)k^*$. Thus k moves to a new higher balanced growth path level. See the figure at right.



As k rises, y – output per worker – also rises. Since a constant fraction of output is saved, c – consumption per worker – rises as y rises. This is summarized in the figures below.



(b) By definition, output can be written as $Y \equiv Ly$. Thus the growth rate of output is $\dot{Y}/Y = \dot{L}/L + \dot{y}/y$. On the initial balanced growth path, $\dot{y}/y = 0$ – output per worker is constant – so $\dot{Y}/Y = \dot{L}/L = n$. On the final balanced growth path, $\dot{y}/y = 0$ again – output per worker is constant again – and so $\dot{Y}/Y = \dot{L}/L = n_{NEW} < n$. In the end, output will be growing at a permanently lower rate.



What happens during the transition? Examine the production function Y = F(K,AL). On the initial balanced growth path AL, K and thus Y are all growing at rate n. Then suddenly AL begins growing at some new lower rate n_{NEW}. Thus suddenly Y will be growing at some rate between that of K (which is growing at n) and that of AL (which is growing at n_{NEW}). Thus, during the transition, output grows more rapidly than it will on the new balanced growth path, but less rapidly than it would have without the decrease in population growth. As output growth gradually slows down during the transition, so does capital growth until finally K, AL, and thus Y are all growing at the new lower n_{NEW}.

<u>Problem 1.7</u>

The derivative of y* = f(k*) with respect to n is given by

(1) $\partial y^*/\partial n = f'(k^*)[\partial k^*/\partial n]$.

To find $\partial k^*/\partial n$, use the equation for the evolution of the capital stock per unit of effective labor, $\dot{k} = sf(k) - (n + g + \delta)k$. In addition, use the fact that on a balanced growth path, $\dot{k} = 0$, k = k* and thus sf(k*) = (n + g + δ)k*. Taking the derivative of both sides of this expression with respect to n yields

(2) $sf'(k^*)\dfrac{\partial k^*}{\partial n} = (n + g + \delta)\dfrac{\partial k^*}{\partial n} + k^*$,

and rearranging yields

(3) $\dfrac{\partial k^*}{\partial n} = \dfrac{k^*}{sf'(k^*) - (n + g + \delta)}$.

Substituting equation (3) into equation (1) gives us

(4) $\dfrac{\partial y^*}{\partial n} = f'(k^*)\left[\dfrac{k^*}{sf'(k^*) - (n + g + \delta)}\right]$.

Rearranging the condition that implicitly defines k*, sf(k*) = (n + g + δ)k*, and solving for s yields
(5) $s = (n + g + \delta)k^*/f(k^*)$.
Substitute equation (5) into equation (4):
(6) $$\frac{\partial y^*}{\partial n} = \frac{f'(k^*)k^*}{[(n+g+\delta)f'(k^*)k^*/f(k^*)] - (n+g+\delta)}.$$
To turn this into the elasticity that we want, multiply both sides of equation (6) by n/y*:
(7) $$\frac{n}{y^*}\frac{\partial y^*}{\partial n} = \frac{n}{(n+g+\delta)}\frac{f'(k^*)k^*/f(k^*)}{[f'(k^*)k^*/f(k^*)] - 1}.$$
Using the definition that $\alpha_K(k^*) \equiv f'(k^*)k^*/f(k^*)$ gives us
(8) $$\frac{n}{y^*}\frac{\partial y^*}{\partial n} = -\frac{n}{(n+g+\delta)}\left[\frac{\alpha_K(k^*)}{1-\alpha_K(k^*)}\right].$$

Now, with $\alpha_K(k^*) = 1/3$, g = 2% and δ = 3%, we need to calculate the effect on y* of a fall in n from 2% to 1%. Using the midpoint of n = 0.015 to calculate the elasticity gives us
(9) $$\frac{n}{y^*}\frac{\partial y^*}{\partial n} = -\frac{0.015}{(0.015+0.02+0.03)}\left(\frac{1/3}{1-1/3}\right) \cong -0.12.$$
So this 50% drop in the population growth rate, from 2% to 1%, will lead to approximately a 6% increase in the level of output per unit of effective labor, since (-0.50)(-0.12) = 0.06. This calculation illustrates the point that observed differences in population growth rates across countries are not nearly enough to account for differences in y that we see.

Problem 1.8

(a) A permanent increase in the fraction of output that is devoted to investment from 0.15 to 0.18 represents a 20 percent increase in the saving rate. From equation (1.27) in the text, the elasticity of output with respect to the saving rate is
(1) $$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1-\alpha_K(k^*)},$$
where $\alpha_K(k^*)$ is the share of income paid to capital (assuming that capital is paid its marginal product).

Substituting the assumption that $\alpha_K(k^*) = 1/3$ into equation (1) gives us
(2) $$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1-\alpha_K(k^*)} = \frac{1/3}{1-1/3} = \frac{1}{2}.$$
Thus the elasticity of output with respect to the saving rate is 1/2. So this 20 percent increase in the saving rate – from s = 0.15 to $s_{NEW} = 0.18$ – causes output to rise relative to what it would have been by about 10 percent. [Note that the analysis has been carried out in terms of output per unit of effective labor. Since the paths of A and L are not affected, however, if output per unit of effective labor rises by 10 percent, output itself is also 10 percent higher than what it would have been.]

(b) Consumption rises less than output. Output ends up 10 percent higher than what it would have been. But the fact that the saving rate is higher means that we are now consuming a smaller fraction of output. We can calculate the elasticity of consumption with respect to the saving rate. On the balanced growth path, consumption is given by
(3) $c^* = (1 - s)y^*$.
Taking the derivative with respect to s yields

(4) $\dfrac{\partial c^*}{\partial s} = -y^* + (1-s)\dfrac{\partial y^*}{\partial s}$.

To turn this into an elasticity, multiply both sides of equation (4) by s/c*:

(5) $\dfrac{\partial c^*}{\partial s}\dfrac{s}{c^*} = \dfrac{-y^* s}{(1-s)y^*} + (1-s)\dfrac{\partial y^*}{\partial s}\dfrac{s}{(1-s)y^*}$,

where we have substituted c* = (1 - s)y* on the right-hand side. Simplifying gives us

(6) $\dfrac{\partial c^*}{\partial s}\dfrac{s}{c^*} = \dfrac{-s}{(1-s)} + \dfrac{\partial y^*}{\partial s}\dfrac{s}{(1-s)y^*}$.

From part (a), the second term on the right-hand side of (6), the elasticity of output with respect to the saving rate, equals 1/2. We can use the midpoint between s = 0.15 and s_{NEW} = 0.18 to calculate the elasticity:

(7) $\dfrac{\partial c^*}{\partial s}\dfrac{s}{c^*} = \dfrac{-0.165}{(1-0.165)} + 0.5 \cong 0.30$.

Thus the elasticity of consumption with respect to the saving rate is approximately 0.3. So this 20% increase in the saving rate will cause consumption to be approximately 6% above what it would have been.

(c) The immediate effect of the rise in investment as a fraction of output is that consumption falls. Although y* does not jump immediately – it only begins to move toward its new, higher balanced-growth-path level – we are now saving a greater fraction, and thus consuming a smaller fraction, of this same y*. At the moment of the rise in s by 3 percentage points – since c = (1 - s)y* and y* is unchanged – c falls. In fact, the percentage change in c will be the percentage change in (1 - s). Now, (1 - s) falls from 0.85 to 0.82, which is approximately a 3.5 percent drop. Thus at the moment of the rise in s, consumption falls by about three and a half percent.

We can use some results from the text on the speed of convergence to determine the length of time it takes for consumption to return to what it would have been without the increase in the saving rate. After the initial rise in s, s remains constant throughout. Since c = (1 - s)y, this means that consumption will grow at the same rate as y on the way to the new balanced growth path. In the text it is shown that the rate of convergence of k and y, after a linear approximation, is given by $\lambda = (1 - \alpha_K)(n + g + \delta)$. With $(n + g + \delta)$ equal to 6 percent per year and $\alpha_K = 1/3$, this yields a value for λ of about 4 percent. This means that k and y move about 4 percent of the remaining distance toward their balanced-growth-path values of k* and y* each year. Since c is proportional to y, c = (1 - s)y, it also approaches its new balanced-growth-path value at that same constant rate. That is, analogous to equation (1.31) in the text, we could write

(8) $c(t) - c^* \cong e^{-(1-\alpha_K)(n+g+\delta)t}[c(0) - c^*]$,

or equivalently

(9) $e^{-\lambda t} = \dfrac{c(t) - c^*}{c(0) - c^*}$.

The term on the right-hand side of equation (9) is the fraction of the distance to the balanced growth path that remains to be traveled.

We know that consumption falls initially by 3.5 percent and will eventually be 6 percent higher than it would have been. Thus it must change by 9.5 percent on the way to the balanced growth path. It will therefore be equal to what it would have been about 36.8 percent (3.5%/9.5% ≅ 36.8%) of the way to the new balanced growth path. Equivalently, this is when the remaining distance to the new balanced growth path is 63.2 percent of the original distance. In order to determine the length of time this will take, we need to find a t* that solves

(10) $e^{-\lambda t^*} = 0.632$.

Taking the natural logarithm of both sides of equation (10) yields

(11) $-\lambda t^* = \ln(0.632)$.

Rearranging to solve for t gives us

(12) $t^* = 0.459/0.04$,

and thus

(13) $t^* \cong 11.5$ years.

It will take a fairly long time – over a decade – for consumption to return to what it would have been in the absence of the increase in investment as a fraction of output.

Problem 1.9

(a) Define the marginal product of labor to be $w \equiv \partial F(K,AL)/\partial L$. Then write the production function as $Y = ALf(k) = ALf(K/AL)$. Taking the partial derivative of output with respect to L yields

(1) $w \equiv \partial Y/\partial L = ALf'(k)[-K/AL^2] + Af(k) = A[(-K/AL)f'(k) + f(k)] = A[f(k) - kf'(k)]$,

as required.

(b) Define the marginal product of capital as $r \equiv [\partial F(K,AL)/\partial K] - \delta$. Again, writing the production function as $Y = ALf(k) = ALf(K/AL)$ and now taking the partial derivative of output with respect to K yields

(2) $r \equiv [\partial Y/\partial K] - \delta = ALf'(k)[1/AL] - \delta = f'(k) - \delta$.

Substitute equations (1) and (2) into wL + rK:

(3) $wL + rK = A[f(k) - kf'(k)]L + [f'(k) - \delta]K = ALf(k) - f'(k)[K/AL]AL + f'(k)K - \delta K$.

Simplifying gives us

(4) $wL + rK = ALf(k) - f'(k)K + f'(k)K - \delta K = Alf(k) - \delta K \equiv ALF(K/AL, 1) - \delta K$.

Finally, since F is constant returns to scale, equation (4) can be rewritten as

(5) $wL + rK = F(ALK/AL, AL) - \delta K = F(K, AL) - \delta K$.

(c) As shown above, $r = f'(k) - \delta$. Since δ is a constant and since k is constant on a balanced growth path, so is $f'(k)$ and thus so is r. In other words, on a balanced growth path, $\dot{r}/r = 0$. Thus the Solow model does exhibit the property that the return to capital is constant over time.

Since capital is paid its marginal product, the share of output going to capital is rK/Y. On a balanced growth path,

(6) $$\frac{\dot{(rK/Y)}}{(rK/Y)} = \dot{r}/r + \dot{K}/K - \dot{Y}/Y = 0 + (n+g) - (n+g) = 0.$$

Thus, on a balanced growth path, the share of output going to capital is constant. Since the shares of output going to capital and labor sum to one, this implies that the share of output going to labor is also constant on the balanced growth path.

We need to determine the growth rate of the marginal product of labor, w, on a balanced growth path. As shown above, $w = A[f(k) - kf'(k)]$. Taking the time derivative of the log of this expression yields the growth rate of the marginal product of labor:

(7) $$\frac{\dot{w}}{w} = \frac{\dot{A}}{A} + \frac{\dot{[f(k) - kf'(k)]}}{[f(k) - kf'(k)]} = g + \frac{[f'(k)\dot{k} - \dot{k}f'(k) - kf''(k)\dot{k}]}{f(k) - kf'(k)} = g + \frac{-kf''(k)\dot{k}}{f(k) - kf'(k)}.$$

On a balanced growth path $\dot{k} = 0$ and so $\dot{w}/w = g$. That is, on a balanced growth path, the marginal product of labor rises at the rate of growth of the effectiveness of labor.

(d) As shown in part (c), the growth rate of the marginal product of labor is

(8) $$\frac{\dot{w}}{w} = g + \frac{-kf''(k)\dot{k}}{f(k) - kf'(k)}.$$

If k < k*, then as k moves toward k*, $\dot{w}/w > g$. This is true because the denominator of the second term on the right-hand side of equation (8) is positive because f(k) is a concave function. The numerator of that same term is positive because k and $\dot{k}$ are positive and f '' (k) is negative. Thus, as k rises toward k*, the marginal product of labor grows faster than on the balanced growth path. Intuitively, the marginal product of labor rises by the rate of growth of the effectiveness of labor on the balanced growth path. As we move from k to k*, however, the amount of capital per unit of effective labor is also rising which also makes labor more productive and this increases the marginal product of labor even more.

The growth rate of the marginal product of capital, r, is

(9) $$\frac{\dot{r}}{r} = \frac{\dot{[f'(k)]}}{f'(k)} = \frac{f''(k)\dot{k}}{f'(k)}.$$

As k rises toward k*, this growth rate is negative since f ' (k) > 0, f '' (k) < 0 and $\dot{k}$ > 0. Thus, as the economy moves from k to k*, the marginal product of capital falls. That is, it grows at a rate less than on the balanced growth path where its growth rate is 0.

Problem 1.10

(a) By definition a balanced growth path occurs when all the variables of the model are growing at constant rates. Despite the differences between this model and the usual Solow model, it turns out that we can again show that the economy will converge to a balanced growth path by examining the behavior of k ≡ K/AL.

Taking the time derivative of both sides of the definition of k ≡ K/AL gives us

(1) $$\dot{k} = \dot{\left(\frac{K}{AL}\right)} = \frac{\dot{K}(AL) - K[\dot{L}A - \dot{A}L]}{(AL)^2} = \frac{\dot{K}}{AL} - \frac{K}{AL}\left[\frac{\dot{L}A + \dot{A}L}{AL}\right] = \frac{\dot{K}}{AL} - k\left(\frac{\dot{L}}{L} + \frac{\dot{A}}{A}\right).$$

Substituting the capital-accumulation equation, $\dot{K} = [\partial F(K, AL)/\partial K]K - \delta K$, and the constant growth rates of the labor force and technology, $\dot{L}/L = n$ and $\dot{A}/A = g$, into equation (1) yields

(2) $$\dot{k} = \frac{[\partial F(K, AL)/\partial K]\,K - \delta K}{AL} - (n+g)k = \frac{\partial F(K, AL)}{\partial K}k - \delta k - (n+g)k.$$

Substituting ∂F(K,AL)/∂K = f '(k) into equation (2) gives us $\dot{k} = f'(k)k - \delta k - (n+g)k$ or simply

(3) $$\dot{k} = [f'(k) - (n+g+\delta)]k.$$

Capital per unit of effective labor will be constant when $\dot{k} = 0$, i.e. when [f ' (k) - (n + g + δ)] k = 0. This condition holds if k = 0 (a case we will ignore) or f ' (k) - (n + g + δ) = 0. Thus the balanced-growth-path level of the capital stock per unit of effective labor is implicitly defined by f '(k*) = (n + g + δ). Since capital per unit of effective labor, k ≡ K/AL, is constant on the balanced growth path, K must grow at the same rate as AL, which grows at rate n + g. Since the production function has constant returns to capital and effective labor, which both grow at rate n + g on the balanced growth path, output must also grow at rate n + g on the balanced growth path. Thus we have found a balanced growth path where all the variables of the model grow at constant rates.

The next step is to show that the economy actually converges to this balanced growth path. At k = k*, f ' (k) = (n + g +δ). If k > k*, f ' (k) < (n + g + δ). This follows from the assumption that f '' (k) < 0 which means that f ' (k) falls as k rises. Thus if k > k*, we have $\dot{k} < 0$ so that k will fall toward its balanced-growth-path value. If k < k*, f ' (k) > (n + g + δ). Again, this follows from the assumption that f '' (k) < 0 which means that f ' (k) rises as k falls. Thus if k < k*, we have $\dot{k} > 0$ so that k will rise toward its balanced-growth-path value. Thus, regardless of the initial value of k (as long as it is not zero), the economy will converge to a balanced growth path at k*, where all the variables in the model are growing at constant rates.

(b) The golden-rule level of k – the level of k that maximizes consumption per unit of effective labor – is defined implicitly by $f'(k^{GR}) = (n + g + \delta)$. This occurs when the slope of the production function equals the slope of the break-even investment line. Note that this is exactly the level of k that the economy converges to in this model where all capital income is saved and all labor income is consumed.

In this model, we are saving capital's contribution to output, which is the marginal product of capital times the amount of capital. If that contribution exceeds break-even investment, (n + g + δ)k, then k rises. If it is less than break-even investment, k falls. Thus k settles down to a point where saving, the marginal product of capital times k, equals break-even investment, (n + g + δ)k. That is, the economy settles down to a point where f ' (k)k = (n + g + δ)k or equivalently f ' (k) = (n + g + δ).

<u>Problem 1.11</u>

We know that $\dot{y}$ is determined by k but since k = g(y), where $g(\bullet) = f^{-1}(\bullet)$, we can write $\dot{y} = \dot{y}(y)$. When k = k* and thus y = y*, $\dot{y} = 0$. A first-order Taylor-series approximation of $\dot{y}(y)$ around y = y* therefore yields

(1) $\dot{y} \cong \left[\left. \frac{\partial \dot{y}}{\partial y} \right|_{y=y^*} \right] (y - y^*)$.

Let λ denote $-\partial \dot{y}(y)/\partial y\big|_{y=y^*}$. With this definition, equation (1) becomes

(2) $\dot{y}(t) \cong -\lambda[y(t) - y^*]$.

Equation (2) implies that in the vicinity of the balanced growth path, y moves toward y* at a speed approximately proportional to its distance from y*. That is, the growth rate of y(t) – y* is approximately constant and equal to -λ. This implies

(3) $y(t) \cong y^* + e^{-\lambda t}[y(0) - y^*]$,

where y(0) is the initial value of y. We now need to determine λ.

Taking the time derivative of both sides of the production function,

(4) y = f(k),

yields

(5) $\dot{y} = f'(k)\dot{k}$.

The equation of motion for capital is given by

(6) $\dot{k} = sf(k) - (n + g + \delta)k$.

Substituting equation (6) into equation (5) yields

(7) $\dot{y} = f'(k)[sf(k) - (n + g + \delta)k]$.

Equation (7) expresses $\dot{y}$ in terms of k. But k = g(y) where $g(\bullet) = f^{-1}(\bullet)$. Thus we can write

(8) $\left.\frac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = \left[\left.\frac{\partial \dot{y}}{\partial k}\right|_{y=y^*}\right]\left[\left.\frac{\partial k}{\partial y}\right|_{y=y^*}\right].$

Taking the derivative of $\dot{y}$ with respect to k gives us

(9) $\frac{\partial \dot{y}}{\partial k} = f''(k)[sf(k) - (n+g+\delta)k] + f'(k)[sf'(k) - (n+g+\delta)]$.

On the balanced growth path, sf(k*) = (n + g + δ)k* and thus

(10) $\left.\frac{\partial \dot{y}}{\partial k}\right|_{y=y^*} = f'(k^*)[sf'(k^*) - (n+g+\delta)]$.

Now, since k = g(y) where g(•) = f^{-1}(•),

(11) $\left.\frac{\partial k}{\partial y}\right|_{y=y^*} = \frac{1}{\left.\frac{\partial y}{\partial k}\right|_{y=y^*}} = \frac{1}{f'(k^*)}$.

Substituting equations (10) and (11) into equation (8) yields

(12) $\left.\frac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = f'(k^*)[sf'(k^*) - (n+g+\delta)]\frac{1}{f'(k^*)}$,

or simply

(13) $\left.\frac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = sf'(k^*) - (n+g+\delta)$.

And thus

(14) $\lambda \equiv -\left.\frac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = (n+g+\delta) - sf'(k^*)$.

Since s = (n + g + δ)k*/f(k*) on the balanced growth path, we can rewrite (14) as

(15) $\lambda \equiv -\left.\frac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = (n+g+\delta) - \frac{(n+g+\delta)k^* f'(k^*)}{f(k^*)}$.

Now use the definition that $\alpha_K \equiv kf'(k)/f(k)$ to rewrite (15) as

(16) $\lambda \equiv -\left.\frac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = [1 - \alpha_K(k^*)](n+g+\delta)$.

Thus y converges to its balanced-growth-path value at rate $[1 - \alpha_K(k^*)](n+g+\delta)$, the same rate at which k converges to its balanced-growth-path value.

Problem 1.12

(a) The production function with capital-augmenting technological progress is given by

(1) $Y(t) = [A(t)K(t)]^{\alpha} L(t)^{1-\alpha}$.

Dividing both sides of equation (1) by $A(t)^{\alpha/(1-\alpha)}L(t)$ yields

(2) $\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{A(t)K(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{\alpha}\left[\frac{L(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{1-\alpha}$,

and simplifying:

(3) $$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{A(t)^{1-\alpha/(1-\alpha)}K(t)}{L(t)}\right]^{\alpha} A(t)^{-\alpha} = \left[\frac{A(t)^{1-\alpha/(1-\alpha)}A(t)^{-1}K(t)}{L(t)}\right]^{\alpha},$$
and thus finally

(4) $$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{K(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{\alpha}.$$

Now, defining $\phi \equiv \alpha/(1 - \alpha)$, $k(t) \equiv K(t)/A(t)^{\phi}L(t)$ and $y(t) \equiv Y(t)/A(t)^{\phi}L(t)$ yields

(5) $y(t) = k(t)^{\alpha}$.

In order to analyze the dynamics of k(t), take the time derivative of both sides of $k(t) \equiv K(t)/A(t)^{\phi}L(t)$:

(6) $$\dot{k}(t) = \frac{\dot{K}(t)\left[A(t)^{\phi}L(t)\right] - K(t)\left[\phi A(t)^{\phi-1}\dot{A}(t)L(t) + \dot{L}(t)A(t)^{\phi}\right]}{\left[A(t)^{\phi}L(t)\right]^{2}},$$

(7) $$\dot{k}(t) = \frac{\dot{K}(t)}{A(t)^{\phi}L(t)} - \frac{K(t)}{A(t)^{\phi}L(t)}\left[\phi\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right],$$

and then using $k(t) \equiv K(t)/A(t)^{\phi}L(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{L}(t)/L(t) = n$ yields

(8) $\dot{k}(t) = \dot{K}(t)/A(t)^{\phi}L(t) - (\phi\mu + n)k(t)$.

The evolution of the total capital stock is given by the usual

(9) $\dot{K}(t) = sY(t) - \delta K(t)$.

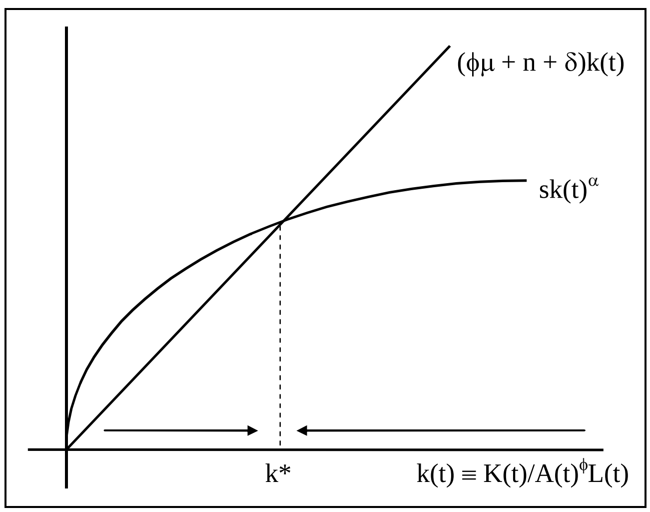
Substituting equation (9) into (8) gives us

(10) $\dot{k}(t) = sY(t)/A(t)^{\phi}L(t) - \delta K(t)/A(t)^{\phi}L(t) - (\phi\mu + n)k(t) = sy(t) - (\phi\mu + n + \delta)k(t)$.

Finally, using equation (5), $y(t) = k(t)^{\alpha}$, we have

(11) $\dot{k}(t) = sk(t)^{\alpha} - (\phi\mu + n + \delta)k(t)$.

Equation (11) is very similar to the basic equation governing the dynamics of the Solow model with labor-augmenting technological progress. Here, however, we are measuring in units of $A(t)^{\phi}L(t)$ rather than in units of effective labor, A(t)L(t). Using the same graphical technique as with the basic Solow model, we can graph both components of $\dot{k}(t)$. See the figure at right.



When actual investment per unit of $A(t)^{\phi}L(t)$, $sk(t)^{\alpha}$, exceeds break-even investment per unit of $A(t)^{\phi}L(t)$, given by $(\phi\mu + n + \delta)k(t)$, k will rise toward k*. When actual investment per unit of $A(t)^{\phi}L(t)$ falls short of break-even investment per unit of $A(t)^{\phi}L(t)$, k will fall toward k*. Ignoring the case in which the initial level of k is zero, the economy will converge to a situation in which k is constant at k*. Since $y = k^{\alpha}$, y will also be constant when the economy converges to k*.

The total capital stock, K, can be written as $A^{\phi}Lk$. Thus when k is constant, K will be growing at the constant rate of $\phi\mu + n$. Similarly, total output, Y, can be written as $A^{\phi}Ly$. Thus when y is constant, output grows at the constant rate of $\phi\mu + n$ as well. Since L and A grow at constant rates by assumption, we have found a balanced growth path where all the variables of the model grow at constant rates.

(b) The production function is now given by

(12) $Y(t) = J(t)^{\alpha} L(t)^{1-\alpha}$.

Define $\bar{J}(t) \equiv J(t)/A(t)$. The production function can then be written as

(13) $Y(t) = \left[A(t)\bar{J}(t)\right]^{\alpha} L(t)^{1-\alpha}$.

Proceed as in part (a). Divide both sides of equation (13) by $A(t)^{\alpha/(1-\alpha)}L(t)$ and simplify to obtain

(14) $$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{\bar{J}(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{\alpha}.$$

Now, defining $\phi \equiv \alpha/(1-\alpha)$, $\bar{j}(t) \equiv \bar{J}(t)/A(t)^{\phi}L(t)$ and $y(t) \equiv Y(t)/A(t)^{\phi}L(t)$ yields

(15) $y(t) = \bar{j}(t)^{\alpha}$.

In order to analyze the dynamics of $\bar{j}(t)$, take the time derivative of both sides of $\bar{j}(t) \equiv \bar{J}(t)/A(t)^{\phi}L(t)$:

(16) $$\dot{\bar{j}} = \frac{\dot{\bar{J}}(t)\left[A(t)^{\phi}L(t)\right] - \bar{J}(t)\left[\phi A(t)^{\phi-1}\dot{A}(t)L(t) + \dot{L}(t)A(t)^{\phi}\right]}{\left[A(t)^{\phi}L(t)\right]^2},$$

(17) $$\dot{\bar{j}}(t) = \frac{\dot{\bar{J}}(t)}{A(t)^{\phi}L(t)} - \frac{\bar{J}(t)}{A(t)^{\phi}L(t)}\left[\phi\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right],$$

and then using $\bar{j}(t) \equiv \bar{J}(t)/A(t)^{\phi}L(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{L}(t)/L(t) = n$ yields

(18) $\dot{\bar{j}}(t) = \dot{\bar{J}}(t)/A(t)^{\phi}L(t) - (\phi\mu + n)\bar{j}(t)$.

The next step is to get an expression for $\dot{\bar{J}}(t)$. Take the time derivative of both sides of $\bar{J}(t) \equiv J(t)/A(t)$:

(19) $$\dot{\bar{J}}(t) = \frac{\dot{J}(t)A(t) - J(t)\dot{A}(t)}{A(t)^2} = \frac{\dot{J}(t)}{A(t)} - \frac{\dot{A}(t)}{A(t)}\frac{J(t)}{A(t)}.$$

Now use $\bar{J}(t) \equiv J(t)/A(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{J}(t) = sA(t)Y(t) - \delta J(t)$ to obtain

(20) $$\dot{J}(t) = \frac{sA(t)Y(t)}{A(t)} - \frac{\delta J(t)}{A(t)} - \mu\bar{J}(t),$$

or simply

(21) $\dot{\bar{J}}(t) = sY(t) - (\mu + \delta)\bar{J}(t)$.

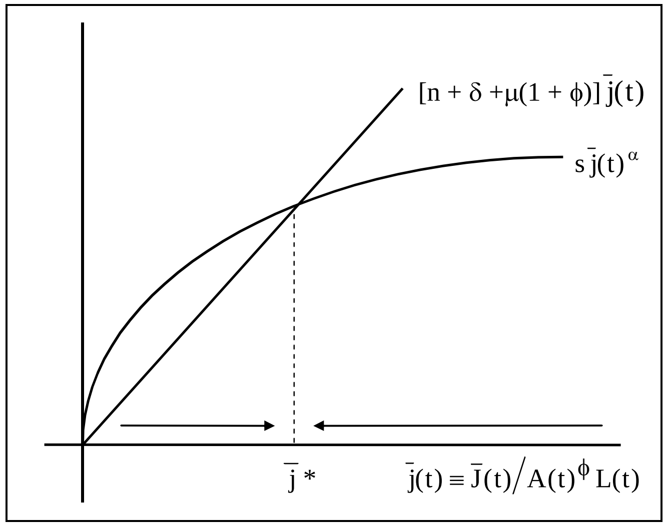
Substitute equation (21) into equation (18):

(22) $\dot{\bar{j}}(t) = sY(t)/A(t)^{\phi}L(t) - (\mu + \delta)\bar{J}(t)/A(t)^{\phi}L(t) - (\phi\mu + n)\bar{j}(t) = sy(t) - \left[n + \delta + \mu(1+\phi)\right]\bar{j}(t)$.

Finally, using equation (15), $y(t) = \bar{j}(t)^{\alpha}$, we have

(23) $\dot{\bar{j}}(t) = s\bar{j}(t)^{\alpha} - \left[n + \delta + \mu(1+\phi)\right]\bar{j}(t)$.

Using the same graphical technique as in the basic Solow model, we can graph both components of $\dot{\bar{j}}(t)$.



See the figure at right. Ignoring the possibility that the initial value of $\bar{j}$ is zero, the economy will converge to a situation where $\bar{j}$ is constant at $\bar{j}^*$. Since $y = \bar{j}^{\alpha}$, y will also be constant when the economy converges to $\bar{j}^*$.

The level of total output, Y, can be written as $A^{\phi}Ly$. Thus when y is constant, output grows at the constant rate of $\phi\mu + n$.

By definition, $\bar{J} \equiv A^{\phi}L\bar{j}$. Once the economy converges to the situation where $\bar{j}$ is constant, $\bar{J}$ grows at the constant rate of $\phi\mu + n$. Since $J \equiv \bar{J}A$, the effective capital stock, J, grows at rate $\phi\mu + n + \mu$ or $n + \mu(1 + \phi)$. Thus the economy does converge to a balanced growth path where all the variables of the model are growing at constant rates.

(c) On the balanced growth path, $\dot{\bar{j}}(t) = 0$ and thus from equation (23):

(24) $s\bar{j}^{\alpha} = [n + \delta + \mu(1+\phi)]\bar{j} \quad \Rightarrow \quad \bar{j}^{1-\alpha} = s/[n + \delta + \mu(1+\phi)]$,

and thus

(25) $\bar{j}^* = \left[s/\left(n + \delta + \mu(1+\phi)\right)\right]^{1/(1-\alpha)}$.

Substitute equation (25) into equation (15) to get an expression for output per unit of $A(t)^{\phi}L(t)$ on the balanced growth path:

(26) $y^* = \left[s/\left(n + \delta + \mu(1+\phi)\right)\right]^{\alpha/(1-\alpha)}$.

Take the derivative of y* with respect to s:

(27) $\dfrac{\partial y^*}{\partial s} = \left[\dfrac{\alpha}{1-\alpha}\right]\left[\dfrac{s}{n + \delta + \mu(1+\phi)}\right]^{\alpha/(1-\alpha)-1}\left[\dfrac{1}{n + \delta + \mu(1+\phi)}\right]$.

In order to turn this into an elasticity, multiply both sides by s/y* using the expression for y* from equation (26) on the right-hand side:

(28) $\dfrac{\partial y^*}{\partial s}\dfrac{s}{y^*} = \left[\dfrac{\alpha}{1-\alpha}\right]\left[\dfrac{s}{n + \delta + \mu(1+\phi)}\right]^{\alpha/(1-\alpha)-1}\left[\dfrac{1}{n + \delta + \mu(1+\phi)}\right]s\left[\dfrac{s}{n + \delta + \mu(1+\phi)}\right]^{-\alpha/(1-\alpha)}$.

Simplifying yields

(29) $\dfrac{\partial y^*}{\partial s}\dfrac{s}{y^*} = \left[\dfrac{\alpha}{1-\alpha}\right]\left[\dfrac{n + \delta + \mu(1+\phi)}{s}\right]\left[\dfrac{s}{n + \delta + \mu(1+\phi)}\right]$,

and thus finally

(30) $\dfrac{\partial y^*}{\partial s}\dfrac{s}{y^*} = \dfrac{\alpha}{1-\alpha}$.

(d) A first-order Taylor approximation of $\dot{y}$ around the balanced-growth-path value of $y = y^*$ will be of the form

(31) $\dot{y} \cong \partial\dot{y}/\partial y\big|_{y=y^*}\left[y - y^*\right]$.

Taking the time derivative of both sides of equation (15) yields

(32) $\dot{y} = \alpha\bar{j}^{\alpha-1}\dot{\bar{j}}$.

Substitute equation (23) into equation (32):

(33) $\dot{y} = \alpha\bar{j}^{\alpha-1}\left[s\bar{j}^{\alpha} - \left(n+\delta+\mu(1+\phi)\right)\bar{j}\right]$,

or

(34) $\dot{y} = s\alpha\bar{j}^{2\alpha-1} - \alpha\bar{j}^{\alpha}\left[n+\delta+\mu(1+\phi)\right]$.

Equation (34) expresses $\dot{y}$ in terms of $\bar{j}$. We can express $\bar{j}$ in terms of y: since y = $\bar{j}^{\alpha}$, we can write $\bar{j}$ = $y^{1/\alpha}$. Thus $\partial\dot{y}/\partial y$ evaluated at y = y* is given by

(35) $$\left.\frac{\partial\dot{y}}{\partial y}\right|_{y=y^*} = \left[\left.\frac{\partial\dot{y}}{\partial\bar{j}}\right|_{y=y^*}\right]\left[\left.\frac{\partial\bar{j}}{\partial y}\right|_{y=y^*}\right] = \left[s\alpha(2\alpha-1)\bar{j}^{2(\alpha-1)} - \alpha^2\bar{j}^{\alpha-1}\left(n+\delta+\mu(1+\phi)\right)\right]\left[\frac{1}{\alpha}y^{(1-\alpha)/\alpha}\right].$$

Now, $y^{(1-\alpha)/\alpha}$ is simply $\bar{j}^{1-\alpha}$ since y = $\bar{j}^{\alpha}$ and thus

(36) $$\left.\frac{\partial\dot{y}}{\partial y}\right|_{y=y^*} = s(2\alpha-1)\bar{j}^{2(\alpha-1)+(1-\alpha)} - \alpha\bar{j}^{\alpha-1+(1-\alpha)}\left[n+\delta+\mu(1+\phi)\right] = s(2\alpha-1)\bar{j}^{\alpha-1} - \alpha\left[n+\delta+\mu(1+\phi)\right].$$

Finally, substitute out for s by rearranging equation (25) to obtain $s = \bar{j}^{1-\alpha}\left[n+\delta+\mu(1+\phi)\right]$ and thus

(37) $$\left.\frac{\partial\dot{y}}{\partial y}\right|_{y=y^*} = \bar{j}^{1-\alpha}\left[n+\delta+\mu(1+\phi)\right](2\alpha-1)\bar{j}^{\alpha-1} - \alpha\left[n+\delta+\mu(1+\phi)\right],$$

or simply

(38) $$\left.\frac{\partial\dot{y}}{\partial y}\right|_{y=y^*} = -(1-\alpha)\left[n+\delta+\mu(1+\phi)\right].$$

Substituting equation (38) into equation (31) gives the first-order Taylor expansion:

(39) $\dot{y} \cong -(1-\alpha)\left[n+\delta+\mu(1+\phi)\right]\left[y - y^*\right]$.

Solving this differential equation (as in the text) yields

(40) $y(t) - y^* = e^{-(1-\alpha)\left[n+\delta+\mu(1+\phi)\right]}\left[y(0) - y^*\right]$.

This means that the economy moves fraction (1 - α)[n + δ + μ(1 + ϕ)] of the remaining distance toward y* each year.

(e) The elasticity of output with respect to s is the same in this model as in the basic Solow model. The speed of convergence is faster in this model. In the basic Solow model, the rate of convergence is given by (1 - α)[n + δ + μ], which is less than the rate of convergence in this model, (1 - α)[n + δ + μ(1 + ϕ)], since $\phi \equiv \alpha/(1-\alpha)$ is positive.

Problem 1.13

(a) The growth-accounting technique of Section 1.7 yields the following expression for the growth rate of output per person:

(1) $$\frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)} = \alpha_K(t)\left[\frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}\right] + R(t),$$

where α_K (t) is the elasticity of output with respect to capital at time t and R(t) is the Solow residual.

Now imagine applying this growth-accounting equation to a Solow economy that is on its balanced growth path. On the balanced growth path, the growth rates of output per worker and capital per worker are both equal to g, the growth rate of A. Thus equation (1) implies that growth accounting would attribute a fraction α_K of growth in output per worker to growth in capital per worker. It would attribute the rest – fraction $(1 - \alpha_K)$ – to technological progress, as this is what would be left in the Solow residual. So with our usual estimate of $\alpha_K = 1/3$, growth accounting would attribute about 67 percent of the growth in output per worker to technological progress and about 33 percent of the growth in output per worker to growth in capital per worker.

(b) In an accounting sense, the result in part (a) would be true, but in a deeper sense it would not: the reason that the capital-labor ratio grows at rate g on the balanced growth path is because the effectiveness of labor is growing at rate g. That is, the growth in the effectiveness of labor – the growth in A – raises output per worker through two channels. First, by directly raising output but also by (for a given saving rate) increasing the resources devoted to capital accumulation and thereby raising the capital-labor ratio. Growth accounting attributes the rise in output per worker through the second channel to growth in the capital-labor ratio, and not to its underlying source. Thus, although growth accounting is often instructive, it is not appropriate to interpret it as shedding light on the underlying determinants of growth.

Problem 1.14

(a) Ordinary least squares (OLS) yields a biased estimate of the slope coefficient of a regression if the explanatory variable is correlated with the error term. We are given that

(1) $\ln\left[\left(Y/N\right)_{1979}\right] - \ln\left[\left(Y/N\right)_{1870}\right]^* = a + b\ln\left[\left(Y/N\right)_{1870}\right]^* + \varepsilon$, and

(2) $\ln\left[\left(Y/N\right)_{1870}\right] = \ln\left[\left(Y/N\right)_{1870}\right]^* + u$,

where ε and u are assumed to be uncorrelated with each other and with the true unobservable 1870 income per person variable, $\ln[(Y/N)_{1870}]^*$.

Substituting equation (2) into (1) and rearranging yields

(3) $\ln\left[\left(Y/N\right)_{1979}\right] - \ln\left[\left(Y/N\right)_{1870}\right] = a + b\ln\left[\left(Y/N\right)_{1870}\right] + \left[\varepsilon - (1+b)u\right]$.

Running an OLS regression on model (3) will yield a biased estimate of b if $\ln[(Y/N)_{1870}]$ is correlated with the error term, $[\varepsilon - (1 + b)u]$. In general, of course, this will be the case since u is the measurement error that helps to determine the value of $\ln[(Y/N)_{1870}]$ that we get to observe. However, in the special case in which the true value of $b = -1$, the error term in model (3) is simply ε. Thus OLS will be unbiased since the explanatory variable will no longer be correlated with the error term.

(b) Measurement error in the dependent variable will not cause a problem for OLS estimation and is, in fact, one of the justifications for the disturbance term in a regression model. Intuitively, if the measurement error is in 1870 income per capita, the explanatory variable, there will be a bias toward finding convergence. If 1870 income per capita is overstated, growth is understated. This looks like convergence: a "high" initial income country growing slowly. Similarly, if 1870 income per capita is understated, growth is overstated. This also looks like convergence: a "low" initial income country growing quickly.

Suppose instead that it is only 1979 income per capita that is subject to random, mean-zero measurement error. When 1979 income is overstated, so is growth for a given level of 1870 income. When 1979 income is understated, so is growth for a given 1870 income. Either case is equally likely: overstating 1979 income for any given 1870 income is just as likely as understating it (or more precisely, measurement error is on average equal to zero). Thus there is no reason for this to systematically cause us to see more or less convergence than there really is in the data.

Problem 1.15

On a balanced growth path, K and Y must be growing at a constant rate. The equation of motion for capital, $\dot{K}(t) = sY(t) - \delta K(t)$, implies the growth rate of K is

(1) $$\frac{\dot{K}(t)}{K(t)} = s\frac{Y(t)}{K(t)} - \delta.$$

As in the model in the text, Y/K must be constant in order for the growth rate of K to be constant. That is, the growth rates of Y and K must be equal.

Taking logs of both sides of the production function, $Y(t) = K(t)^{\alpha} R(t)^{\beta} T(t)^{\gamma} [A(t)L(t)]^{1-\alpha-\beta-\gamma}$, yields

(2) $\ln Y(t) = \alpha \ln K(t) + \beta \ln R(t) + \gamma \ln T(t) + (1 - \alpha - \beta - \gamma)[\ln A(t) + \ln L(t)]$.

Differentiating both sides of (2) with respect to time gives us

(3) $$g_Y(t) = \alpha g_K(t) + \beta g_R(t) + \gamma g_T(t) + (1 - \alpha - \beta - \gamma)\left[g_A(t) + g_L(t)\right].$$

Substituting in the facts that the growth rates of R, T, and L are all equal to n and the growth rate of A is equal to g gives us

(4) $$g_Y(t) = \alpha g_K(t) + \beta n + \gamma n + (1 - \alpha - \beta - \gamma)(n + g).$$

Simplifying gives us

(5) $$\begin{aligned} g_Y(t) &= \alpha g_K(t) + (\beta + \gamma)n + (1 - \alpha)n - (\beta + \gamma)n + (1 - \alpha - \beta - \gamma)g \\ &= \alpha g_K(t) + (1 - \alpha)n + (1 - \alpha - \beta - \gamma)g \end{aligned}$$

Using the fact that g_Y and g_K must be equal on a balanced growth path leaves us with

(6) $g_Y = \alpha g_Y + (1 - \alpha)n + (1 - \alpha - \beta - \gamma)g$,

(7) $(1 - \alpha)g_Y = (1 - \alpha)n + (1 - \alpha - \beta - \gamma)g$,

and thus the growth rate of output on the balanced growth path is given by

(8) $$\tilde{g}_Y^{bgp} = \frac{(1 - \alpha)n + (1 - \alpha - \beta - \gamma)g}{1 - \alpha}.$$

The growth rate of output per worker on the balanced growth path is

(9) $$\tilde{g}_{Y/L}^{bgp} = \tilde{g}_Y^{bgp} - \tilde{g}_L^{bgp}.$$

Using equation (8) and the fact that L grows at rate n, we can write

(10) $$\tilde{g}_{Y/L}^{bgp} = \frac{(1 - \alpha)n + (1 - \alpha - \beta - \gamma)g}{1 - \alpha} - n = \frac{(1 - \alpha)n + (1 - \alpha - \beta - \gamma)g - (1 - \alpha)n}{1 - \alpha}.$$

And thus finally

(11) $$\tilde{g}_{Y/L}^{bgp} = \frac{(1 - \alpha - \beta - \gamma)g}{1 - \alpha}.$$

Equation (11) is identical to equation (1.50) in the text.